The Effect of Certain Types of Small Holes on the Stresses in a Uniformly Loaded Isotropic Plate

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NOTATIONS

x, y	Rectangular coordinates in the z plane.
z	Complex variable $z = x + iy$.
i	$i = (-1)^{1/2}$.
R, β	Polar coordinates in the w plane.
w	Complex variable $w = Re^{i\beta}$.
L	Boundary of the hole in the z plane.
K	Unit circle in the w plane.
n, m	Direction cosines of the outer normal.
X_N, Y_N	Components of surface forces per unit area at a point of the boundary.
$\overset{\frown}{xx}, \overset{\frown}{yy}$	Normal components of stress parallel to the x- and y-axes.
$\overset{\frown}{xy}$	Shearing stress.
U	Biharmonic stress function $U = U(x,y)$.
U_x, U_y	Subscripts x and y denote partial differentiation with respect to the indicated variable.
∇^2	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
∇^4	$\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.

CHAPTER I

INTRODUCTION

This is a mathematical analysis of the stress distributions existing in a large, uniformly loaded isotropic plate weakened by a small hole. We will consider the plate, whose plane is the xy-plane, to be subject only to forces directed parallel to the x- and y-axes. It has been shown that the solution to such problems, when body forces are absent or constant, reduces to the integration of the differential equation[§]

$$\nabla^4 U = 0, \tag{1}$$

having regard to the boundary conditions

$$\left.\begin{aligned} X_N &= n\widehat{xx} + m\widehat{xy} \\ Y_N &= m\widehat{yy} + n\widehat{xy}. \end{aligned}\right\} \tag{2}$$

The stress components are represented in terms of the biharmonic stress function by the set of equations

$$\widehat{xx} = U_{yy}, \quad \widehat{yy} = U_{xx}, \quad \widehat{xy} = -U_{xy}, \tag{3}$$

where subscripts on U denote partial derivatives with respect to the indicated variable.

The boundary curve L of the hole in the z plane, will be taken to be given in terms of the polar angle β

[§] See reference 5 of Bibliography.

in the w plane, by

$$\left.\begin{aligned} x = {} & r'\cos 5\beta + r''\sin 5\beta + t'\cos 2\beta \\ & + t''\sin 2\beta + s'\cos \beta - s''\sin \beta, \\ y = {} & r''\cos 5\beta - r'\sin 5\beta + t''\cos 2\beta \\ & - t'\sin 2\beta + s''\cos \beta + s'\sin \beta. \end{aligned}\right\} \qquad (4)$$

The graph of these equations for various values of r', r'', t', t'', s', and s'', will be discussed later.

We use the mapping function

$$z = F(w) = rw^{-5} + tw^{-2} + sw, \qquad (5)$$

to map the region exterior to L onto the region exterior to the unit circle K in the w plane, where

$$r = r' + ir'', \quad t = t' + it'', \quad s = s' + is''. \qquad (6)$$

To find the biharmonic stress function U we formulate the boundary-value problem in the form of integral equations§, in order to apply the method most often associated with the name of N. I. Mushelišvili.§§ He has shown that if F(w) is rational, then the solution of the integral equations reduces to evaluating integrals of Cauchy's type.

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§ See reference 4 of Bibliography.

§§ See references 7 and 8 of Bibliography.

CHAPTER II

THE HOLE

In our discussion of the hole whose boundary L in the complex z plane is given by Eqs. (4), we first consider the conformal mapping of the complex w plane upon the z plane. For the mapping to be conformal a theorem which we shall call Theorem I must hold.

Theorem I. At each point where a function $F(w)$ is analytic and $F'(w) \neq 0$, the mapping $z = F(w)$ is conformal.

In order that the unit circle in the w plane (hereafter called K) shall map upon L, let

$$z = F(w) = rw^{-5} + tw^{-2} + sw. \qquad (7)$$

In the applications which follow, attention is confined to those problems for which the singularities of $F(w)$ and the zeros of $F'(w)$ lie on the boundary or outside of it. From Eq. (7) we see that the point $w \longrightarrow \infty$ corresponds to the point at infinity in the z plane, and that we have no poles at points in the plate. We note that $F(w)$ is a single-valued analytic function everywhere in the plate. The inverse function $w = g(z)$ is a multiple-valued analytic function, and for any value of z there are in general one or more values of w. Hence we find it necessary to assign such values to r, t, and s, that the branch points in the z plane shall lie on the boundary or outside of it, since in the neighborhood of a branch point the mapping by means of an analytic function ceases to be conformal. A criterion for locating such points is given by the following theorem.

Theorem II. Given a multiple-valued analytic function $w = g(z)$, whose inverse function $z = F(w)$ is single-valued. If the derived function $F'(w)$ has a zero of order h, at a finite point w_0, or a pole of order $h + 2$, then $g(z)$ has a branch point of order h at the corresponding point z_0.§

We can also state that if $F'(w)$ has a pole of order h at $w = \infty$, then $g(z)$ has a branch point of order h at the corresponding point in the z plane.

In applying Theorem II, we do not need to consider the poles for $w = 0$, since we will always use values of w so that $|w| \geqq 1$, in order to stay in the proper branch of the w function. From Eq. (7)

$$F'(w) = -5rw^{-6} - 2tw^{-3} + s. \tag{8}$$

To determine the zero points, we have

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§ For proof of Theorem II, see page 335 of reference 6 of Bibliography.

$$sw^6 - 2tw^3 - 5r = 0,$$

$$w^3 = j^3, \qquad w^3 = k^3,$$

where

$$\left.\begin{aligned} j &= (2s)^{-1/3}\left(2t - \sqrt{4t^2 + 20rs}\right)^{1/3}, \\ \text{and} \quad k &= (2s)^{-1/3}\left(2t + \sqrt{4t^2 + 20rs}\right)^{1/3}. \end{aligned}\right\} \quad (9)$$

The six zero points are

$$\left.\begin{aligned} w_1 &= k, \quad w_2 = pk, \quad w_3 = p^2k, \\ w_4 &= j, \quad w_5 = pj, \quad w_6 = p^2j, \end{aligned}\right\} \quad (10)$$

where 1, p, and p^2, are the cube roots of one. The branch points are obtained by substituting the values of w from Eqs.(10) into Eq.(7). Parameters r, t, and s, will be taken so that

$$|j| \leqq 1, \quad |k| \leqq 1. \qquad (11)$$

When $|j| = 1$, or $|k| = 1$, the boundary curve will have cusps, as in Fig. 2, Appendix I. Since these branch points are the zeros of F'(w), inspection of Eqs.(89) to (95) will show that the stresses become infinite at these points. In the neighborhood of such points a plastic flow will occur which will eliminate the infinite stresses, and our equations are not valid in such a neighborhood. When $|j| < 1$, and $|k| < 1$, the boundary curve has continuous curvature and our solution is valid. The mapping $z = F(w)$ is then conformal for all values of $|w| \geqq 1$.

We now consider some illustrations of various holes. The simplest hole is for $r = t = 0$, $s = s'$, which is a circle. This problem has been previously solved so we omit this case.[§]

CASE ONE. Let $r = 0$, and let t and s be real. Equations (4) reduces to

$$\left.\begin{aligned} x &= t \operatorname{Cos} 2\beta + s \operatorname{Cos} \beta, \\ y &= - t \operatorname{Sin} 2\beta + s \operatorname{Sin} \beta. \end{aligned}\right\} \tag{12}$$

From Eq. (8)

$$F'(w) = - 2tw^{-3} + s. \tag{13}$$

The zero points of $F'(w) = 0$, are

$$w_1 = (2ts^{-1})^{1/3}, \quad w_2 = pw_1, \quad w_3 = p^2 w_1. \tag{14}$$

From Eq.(7)

$$z = F(w) = tw^{-2} + sw. \tag{15}$$

On substituting the values of w from (14) into (15), we obtain the branch points

$$z_1 = \tfrac{3}{2}(2ts^2)^{1/3}, \; z_2 = pz_1, \; z_3 = p^2 z_1. \tag{16}$$

In calculating the above values of z_1, z_2, and z_3, we have made use of the relations

$$pp^2 = p^3 = (p^2)^3 = 1, \quad (p^2)^2 = p,$$

which follow from

[§] See reference 5 of Bibliography.

$$p = -\frac{1}{2} + \frac{i(3)^{1/2}}{2}, \quad p^2 = -\frac{1}{2} - \frac{i(3)^{1/2}}{2}.$$

Setting $t = 1$, $s = 5$, in Eq.(12), we have

$$\left.\begin{aligned} x &= \text{Cos } 2\beta + 5 \text{ Cos } \beta, \\ y &= -\text{ Sin } 2\beta + 5 \text{ Sin } \beta. \end{aligned}\right\} \qquad (17)$$

The curve obtained from Eqs.(17) is given in Appendix I, Fig. 1. The branch points calculated from Eqs.(16) give $z_1 = 5.526$, $Z_2 = -2.763 + 4.786i$, $z_3 = -2.763 - 4.786i$, which lie inside L.

From Eqs.(12), on setting $s = 2$, $t = 1$, we have

$$\left.\begin{aligned} x &= \text{Cos } 2\beta + 2 \text{ Cos } \beta, \\ y &= -\text{ Sin } 2\beta + 2 \text{ Sin } \beta. \end{aligned}\right\} \qquad (18)$$

From Eq.(8)

$$F'(w) = -2w^{-3} + 2. \qquad (19)$$

The zero points of $F'(w) = 0$, are

$$w_1 = 1, \quad w_2 = p, \quad w_3 = p^2. \qquad (20)$$

Substituting the values of w from Eqs.(20) into Eqs.(16) we obtain the branch points

$$z_1 = 3, \quad z_2 = 3p, \quad z_3 = 3p^2. \qquad (21)$$

The branch points in Eq.(21) lie on L, since $k = 1$ in Eq.(11). The curve obtained from Eqs.(18) is given in Appendix I, Fig. 2.

CASE TWO. Let $t = 0$, and let r and s be real. Eqs.(4) reduces to

$$\left.\begin{aligned} x &= r \cos 5\beta + s \cos \beta, \\ y &= - r \sin 5\beta + s \sin \beta. \end{aligned}\right\} \tag{22}$$

From Eq. (8)

$$F'(w) = - 5rw^{-6} + s. \tag{23}$$

The six zero points of $F'(w) = 0$, are obtained from Eqs.(9), where

$$k = (5rs^{-1})^{1/6}, \qquad j = - k.$$

From Eq.(7)

$$z = F(w) = rw^{-5} + sw. \tag{24}$$

On substituting these six values of w into Eq.(24), we obtain the branch points

$$z_1 = a,\ z_2 = pa,\ z_3 = p^2a,\ z_4 = - a,\ z_5 = -pa,\ z_6 = -p^2a, \tag{25}$$

where

$$a = \frac{6(rs^5)^{1/6}}{5^{5/6}}. \tag{26}$$

Setting $s = 20$, $r = 1$, Eqs.(22) becomes

$$\left.\begin{aligned} x &= \cos 5\beta + 20 \cos \beta, \\ y &= - \sin 5\beta + 20 \sin \beta. \end{aligned}\right\} \tag{27}$$

The curve obtained from Eqs.(27) is given in Appendix I, Fig. 3. We note that the curve is in the shape of a hexagon where the corners have been rounded off. Substituting $s = 20$, $r = 1$, into Eq.(26), gives $a = (19.044)$. By inspecting Eqs.(25), we see that the branch points lie on a circle of radius a, at intervals of 60°, and are inside of L.

Let $r = 1$, $s = 10$. From Eqs.(22), we obtain

$$\left.\begin{aligned} x &= \operatorname{Cos} 5\beta + 10 \operatorname{Cos} \beta, \\ y &= -\operatorname{Sin} 5\beta + 10 \operatorname{Sin} \beta. \end{aligned}\right\} \qquad (28)$$

The curve obtained from Eqs.(28) is given in Appendix I, Fig. 4. All the branch points lie inside L. There are no sharp corners on the curve.

CASE THREE. This case includes all the holes obtainable from Eqs.(4), which are not already included in Case I or Case II. The stress components given in Eqs.(93) to (95), and the biharmonic stress function U, Eq.(87), are derived for real or complex values of r, s, and t. We shall give two examples of holes that can be obtained from Eqs.(4). Let $r' = r'' = 1$, $t' = t'' = 0$, $s' = s'' = 10$. Then Eqs.(4) reduces to

$$\left.\begin{aligned} x &= \operatorname{Cos} 5\beta + \operatorname{Sin} 5\beta + 10\,(\operatorname{Cos} \beta - \operatorname{Sin} \beta), \\ y &= \operatorname{Cos} 5\beta - \operatorname{Sin} 5\beta + 10\,(\operatorname{Cos} \beta + \operatorname{Sin} \beta). \end{aligned}\right\} \qquad (29)$$

The hole whose boundary is given by Eqs.(29) is given in Appendix I, Fig. 5. Equation (7) reduces to

$$z = F(w) = (1 + i)w^{-5} + 10(1 + i)w. \tag{30}$$

Hence

$$F'(w) = -5(1 + i)w^{-6} + 10(1 + i), \tag{31}$$

and it is seen from Eq.(31), that the zero points of $F'(w) = 0$, are $w^6 = 1/2$. They lie inside K, and the branch points lie inside L.

Let $r' = r'' = 1$, $t' = t'' = 0$, $s' = s'' = 20$. Equation (4) reduces to

$$\left.\begin{aligned} x &= \operatorname{Cos} 5\beta + \operatorname{Sin} 5\beta + 20(\operatorname{Cos} \beta - \operatorname{Sin} \beta), \\ y &= \operatorname{Cos} 5\beta - \operatorname{Sin} 5\beta + 20(\operatorname{Cos} \beta + \operatorname{Sin} \beta). \end{aligned}\right\} \tag{32}$$

In Appendix I, Fig. 6, we have the curve obtained from Eqs.(32). It is similar to the curve in Fig. 3.

CHAPTER III

THE STRESS FUNCTION

The general biharmonic stress function can be expressed in terms of two analytic functions of the complex variable z. Let $\nabla^2 U = P$. From Eq.(1),

$$\nabla^4 U = \nabla^2\nabla^2 U = \nabla^2 P = 0,$$

hence P is a harmonic function. The function P being harmonic, we can construct an analytic function

$$F(z) = P(x,y) + iQ(x,y),$$

where the functions P and Q satisfy the Cauchy-Riemann equations

$$P_x = Q_y, \quad P_y = - Q_x.$$

Let us define an analytic function $\phi(z)$ by the formula

$$\phi(z) = p + iq = 1/4 \int F(z)dz,$$

then

$$\phi'(z) = p_x + iq_x = 1/4F(z) = 1/4(P + iQ),$$

and since $p_x = q_y$, $p_y = - q_x$, we see that

$$p_x = q_y = 1/4P, \quad p_y = - q_x = - 1/4Q.$$

It then follows that

$$\nabla^2(U - xp - yq) = 0,$$

so that the biharmonic function U has the form

$$U = xp + yq + p_0,$$

where p_0 is some harmonic function.

Let us construct the function

$$\int_{z_0}^{z} T(z)dz = p_0 + iq_0,$$

where q_0 is the harmonic conjugate of p_0. The biharmonic function U, in terms of $\phi(z)$ and T(z), is then given by

$$U = \mathrm{Re}(\bar{z}\phi(z) + \int_{z_0}^{z} T(z)dz), \tag{33}$$

where $\bar{z} = x - iy$, and the symbol Re denotes the real part of the expression following it. As a consequence of the last equation, the stresses can be expressed in terms of $\phi(z)$ and T(z). The calculations of the stress components give the following equations.

$$U_y = \mathrm{Re}(-i\phi(z) + i\bar{z}\phi'(z) + iT(z)). \tag{34}$$

$$\begin{aligned} \widehat{xx} &= \mathrm{Re}(2\phi'(z) - \bar{z}\phi''(z) - T'(z)) = U_{yy} \\ &= \phi'(z) + \bar{\phi}'(\bar{z}) - \tfrac{1}{2}(\bar{z}\phi''(z) + z\bar{\phi}''(\bar{z}) + T'(z) + \bar{T}'(\bar{z})). \end{aligned} \tag{35}$$

$$U_x = \mathrm{Re}(\phi(z) + \bar{z}\phi'(z) + T(z)). \tag{36}$$

$$\begin{aligned} \widehat{yy} &= \mathrm{Re}(2\phi'(z) + \bar{z}\phi''(z) + T'(z)) = U_{xx} \\ &= \phi'(z) + \bar{\phi}'(\bar{z}) + \tfrac{1}{2}(\bar{z}\phi''(z) + z\bar{\phi}''(\bar{z}) + T'(z) + \bar{T}'(\bar{z})). \end{aligned} \tag{37}$$

$$\begin{aligned} \widehat{xy} &= -U_{xy} = -\mathrm{Re}(i\bar{z}\phi''(z) + iT'(z)) \\ &= -i\mathrm{I}(\bar{z}\phi''(z) + T'(z)) \\ &= -i\tfrac{1}{2}(\bar{z}\phi''(z) - z\bar{\phi}''(\bar{z}) + T'(z) - \bar{T}'(\bar{z})). \end{aligned} \tag{38}$$

By inspecting Eqs.(35) and (37), we see

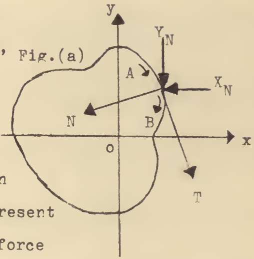
$$\widehat{xx} + \widehat{yy} = 4Re(\phi'(z)) = 2\phi'(z) + 2\bar{\phi}'(\bar{z}). \qquad (39)$$

From Eqs.(35), (37), and (38), we obtain

$$-\widehat{xx} + \widehat{yy} + 2i\widehat{xy} = 2\bar{z}\phi''(z) + 2T'(z). \qquad (40)$$

Consider an arc AB of some curve in the plane of the plate, e.g: the boundary curve. We shall orient the positive direction of the normal N and of the tangent T as indicated in Fig.(a). Let $X_N ds$ and $Y_N ds$ represent the x- and y-components of the force acting on the element of arc ds of AB, from the positive side of the normal N. We have from Eqs.(2)

Fig.(a)



$$\begin{aligned} X_N &= \widehat{xx}\, Cos(x,N) + \widehat{xy}\, Cos(y,N) \\ &= U_{yy} Cos(x,N) - U_{xy} Cos(y,N), \\ Y_N &= \widehat{xy}\, Cos(x,N) + \widehat{yy}\, Cos(y,N) \\ &= U_{xx} Cos(y,N) - U_{xy} Cos(x,N). \end{aligned} \qquad (41)$$

But

$$\begin{aligned} Cos(x,N) &= Cos(y,T) = dy/ds, \\ Cos(y,N) &= - Cos(x,T) = - dx/ds, \end{aligned} \qquad (42)$$

so that

$$X_N = d/ds(U_y), \quad Y_N = - d/ds(U_x). \tag{43}$$

Forming the expression

$$X_N + iY_N = d/ds(U_y - iU_x) = - id/ds(U_x + iU_y),$$

we obtain

$$(X_N + iY_N)ds = - id(U_x + iU_y).$$

Then the quantity

$$\begin{aligned} \int_A^B (X_N + iY_N)ds &= - i\int_A^B d(U_x + iU_y)ds \\ &= - i\Big[\phi(z) + z\bar{\phi}'(\bar{z}) + \bar{T}(\bar{z})\Big]_A^B, \end{aligned} \tag{44}$$

represents the resultant force on the arc AB, where we have substituted the values of U_x and U_y from Eqs.(34) and (36). Equation (44) expresses the boundary conditions in terms of $\phi(z)$ and $T(z)$, when forces are prescribed on a boundary curve. In the case at hand, our plate is subject only to stresses at infinity. Let the prescribed forces be $(\widehat{xx})_\infty = S_1$, $(\widehat{yy})_\infty = S_2$, $(\widehat{xy})_\infty = S_{12}$. Hence, no forces act on the boundary of the hole, and the left hand side of Eq.(44) becomes zero. We shall find it more convenient to deal with these boundary conditions in the w plane. Recalling Eq.(5)

$$z = F(w) = sw + tw^{-2} + rw^{-5}, \tag{45}$$

we have

$$z' = F'(w) = s - 2tw^{-3} - 5rw^{-6}. \tag{46}$$

For the sake of simplicity, we introduce the following notations:

$$\begin{aligned} \phi(z) &\equiv \phi(F(w)) \equiv \phi(w), \\ T(z) &\equiv T(F(w)) \equiv T(w). \end{aligned} \tag{47}$$

Then

$$\phi'(z) = \phi'(w)/F'(w), \tag{48}$$

and

$$T'(z) = T'(w)/F'(w). \tag{49}$$

The value of w on K shall be denoted by v. We have

$$v = e^{i\beta} = (\bar{v})^{-1}. \tag{50}$$

Noting that K corresponds to L, on the boundary

$$\begin{aligned} z\bar{\phi}'(\bar{z}) &= F(v)\bar{\phi}'(\bar{v})\left[\bar{F}'(\bar{v})\right]^{-1} \\ &= \frac{(sv + tv^{-2} + rv^{-5})}{\bar{s} - 2\bar{t}\bar{v}^{-3} - 5\bar{r}\bar{v}^{-6}}\,\bar{\phi}'(\bar{v}) \\ &= v^{-5}\frac{(sv^{6} + tv^{3} + r)}{\bar{s} - 2\bar{t}v^{3} - 5\bar{r}v^{6}}\,\bar{\phi}'(\bar{v}) \\ &= H(v)\bar{\phi}'(\bar{v}). \end{aligned} \tag{51}$$

Substituting Eq.(51) into Eq.(44), the boundary conditions read

$$\phi(v) + H(v)\bar{\phi}'(\bar{v}) + \bar{T}(\bar{v}) = 0, \tag{52}$$

or

$$\bar{\phi}(\bar{v}) + \bar{H}(\bar{v})\phi'(v) + T(v) = 0, \qquad (53)$$

where Eq.(53) is the conjugate of Eq.(52), and

$$\bar{H}(\bar{v}) = v^5(\bar{s} + \bar{t}v^3 + \bar{r}v^6)(sv^6 - 2tv^3 - 5r)^{-1}.$$

It is known that when there is no unbalanced force acting on the hole and when the stresses at infinity are uniform, the functions $\phi(w)$ and $T(w)$ take the form§

$$\phi(w) = Cw + \sum_{n=1}^{\infty} a_n w^{-n}, \qquad (54)$$

$$T(w) = (D + iD')w + \sum_{n=1}^{\infty} b_n w^{-n}, \qquad (55)$$

where the constants C, D, and D', are real, and the constants a_n and b_n may be real or complex. To find the value of the constant C, we apply Eqs.(39), (45), (48), and (54).

$$(\widehat{xx})_\infty + (\widehat{yy})_\infty = (\widehat{xx} + \widehat{yy})_\infty = 2(\phi'(z) + \bar{\phi}'(\bar{z}))_\infty$$

$$= 2(\phi'(w)/F'(w) + \bar{\phi}'(\bar{w})/\bar{F}'(\bar{w}))_\infty$$

$$= 2\left[\frac{C - \sum_{n=1}^{\infty} na_n w^{-n-1}}{s - 2tw^{-3} - 5rw^{-6}} + \frac{C - \sum_{n=1}^{\infty} n\bar{a}_n \bar{w}^{-n-1}}{\bar{s} - 2\bar{t}\bar{w}^{-3} - 5\bar{r}\bar{w}^{-6}}\right]_\infty$$

$$= 4Cs'(s\bar{s})^{-1} = S_1 + S_2.$$

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§ See reference 3 of Bibliography.

To calculate the values of D and D' in Eq.(55), we have from Eq.(40)

$$\begin{aligned}(\widehat{yy})_\infty - (\widehat{xx})_\infty + 2i(\widehat{xy})_\infty &= S_2 - S_1 + 2iS_{12} \\ &= 2(\bar{z}\phi''(z) + T'(z))_\infty .\end{aligned} \tag{56}$$

Taking the derivative of Eq.(48), we obtain

$$\phi''(z)dz/dw = (F'(w))^{-2}\Big[F'(w)\phi''(w) - \phi'(w)F''(w)\Big].$$

Hence

$$\phi''(z) = (F'(w))^{-3}\Big[F'(w)\phi''(w) - \phi'(w)F''(w)\Big]. \tag{57}$$

Substituting the values of $\phi'(w)$, $\phi''(w)$, $F'(w)$, and $F''(w)$, we have

$$\begin{aligned}\phi''(z) = {} & \frac{(s - 2tw^{-3} - 5rw^{-6})(\sum_{n=1}^{\infty} n(n+1)a_n w^{-n-2})}{(s - 2tw^{-3} - 5rw^{-6})^3} \\ & - \frac{(C - \sum_{n=1}^{\infty} na_n w^{-n-1})(6tw^{-4} + 30rw^{-7})}{(s - 2tw^{-3} - 5rw^{-6})^3} .\end{aligned} \tag{58}$$

By inspecting Eq.(58), we see $(\phi''(z))_\infty = 0$. Hence Eq.(56) reduces to

$$\begin{aligned}2(T'(z))_\infty &= \left[\frac{2(D + iD' - \sum_{n=1}^{\infty} nb_n w^{-n-1})}{s - 2tw^{-3} - 5rw^{-6}}\right]_\infty \\ &= 2s^{-1}(D + iD') = S_2 - S_1 + 2iS_{12}.\end{aligned}$$

Let $B = \bar{s}(S_1 + S_2)(4s')^{-1}$, and $B' = 1/2(S_2 - S_1)$.

Then Eqs.(54) and (55) become

$$\emptyset(w) = sBw + \sum_{n=1}^{\infty} a_n w^{-n}, \tag{59}$$

$$T(w) = s(B' + iS_{12})w + \sum_{n=1}^{\infty} b_n w^{-n}. \tag{60}$$

The actual process of solving the problem consists of multiplying Eqs.(52) and (53) by $(2\pi i(v - w))^{-1}dv$ and integrating around K. The integration will be performed with the aid of the following two theorems.

Theorem III. If $f(w)$ is an analytic function within K, except perhaps at $w = 0$, where it has a pole with a principal part of $\sum_{n=1}^{N} A_n w^{-n}$, then $\oint f(v)(2\pi i(v-w))^{-1}dv = -\sum_{n=1}^{N} A_n w^{-n}$, for $|w| > 1$.

To prove Theorem III, we represent $f(w)$ by its Laurent series

$$\sum_{n=1}^{N} A_n w^{-n} + \sum_{m=o}^{\infty} B_m w^m$$

in a neighborhood of the pole $w = 0$. Then applying the Residue Theorem, we have

$$\oint f(v)\left[2\pi i(v-w)\right]^{-1}dv = -(2\pi i)^{-1}\oint f(v)\sum_{p=o}^{\infty} v^p w^{-p-1}dv =$$

$$= - (2\pi i)^{-1} \oint \sum_{p=0}^{\infty} w^{-p-1} \left[\sum_{n=1}^{N} A_n v^{p-n} + \sum_{m=0}^{\infty} B_m v^{m+p} \right] dv$$

$$= - (2\pi i)^{-1} \oint \sum_{n=1}^{N} A_n v^{-1} w^{-n} dv = - \sum_{n=1}^{N} A_n w^{-n}, \text{ for } |w| > 1.$$

Theorem IV. If $f(w)$ is an analytic function outside of K, except perhaps at $w = \infty$, where it has a pole with a principal part of $\sum_{p=0}^{N} A_p w^p$, then

$$\oint f(v)(2\pi i(v - w))^{-1} dv = - f(w) + \sum_{p=0}^{N} A_p w^p, \text{ for } |w| > 1.$$

To prove Theorem IV, we represent $f(w)$ by its Laurent series

$$\sum_{p=0}^{N} A_p w^p + \sum_{m=1}^{\infty} B_m w^{-m}$$

in a neighborhood of the pole $w = \infty$. Then, applying the Residue Theorem, we have

$$\oint f(v)(2\pi i(v - w))^{-1} dv = - (2\pi i)^{-1} \oint f(v) \sum_{n=0}^{\infty} v^n w^{-n-1} dv$$

$$= - (2\pi i)^{-1} \oint \sum_{n=0}^{\infty} w^{-n-1} \left[\sum_{p=0}^{N} A_p v^{p+n} + \sum_{m=1}^{\infty} B_m v^{n-m} \right] dv$$

$$= -(2\pi i)^{-1} \oint \sum_{m=1}^{\infty} B_m v^{-1} w^{-m} dv = - \sum_{m=1}^{\infty} B_m w^{-m}$$

$$= - f(w) + \sum_{p=0}^{N} A_p w^p, \text{ for } |w| > 1.$$

Let us first multiply Eq.(52) by $(2\pi i(v-w))^{-1}dv$ and integrate around K. We have

$$\oint \frac{\phi(v)\, dv}{2\pi i(v-w)} + \oint \frac{H(v)\bar{\phi}'(\bar{v})dv}{2\pi i(v-w)} + \oint \frac{\bar{T}(\bar{v})\, dv}{2\pi i(v-w)} = 0.$$

Applying Theorem III to the first integral in the last equation, we have

$$\begin{aligned} &(2\pi i)^{-1} \oint \frac{(sBv + \sum_{n=1}^{\infty} a_n v^{-n})}{v - w}\, dv \\ &= - \sum_{n=1}^{\infty} a_n w^{-n} = - \phi(w) + sBw. \end{aligned} \tag{61}$$

To evaluate the second integral, we shall write H(v) in the following form.

$$H(v) = r(\bar{s})^{-1}v^{-5} + Ev^{-2} + H_0(v), \tag{62}$$

where the exponents of v in $H_0(v)$ are all positive, and

$$E = (t\bar{s} + 2\bar{t}r)(\bar{s})^{-2}.$$

We note from Theorem IV, that the value of the integral contains only negative powers of w. Hence

$$\begin{aligned} &(2\pi i)^{-1} \oint \frac{H(v)\bar{\phi}'(\bar{v})}{v - w} dv \\ &= (2\pi i)^{-1} \oint (r\bar{s}^{-1}v^{-5} + Ev^{-2})(\bar{s}\bar{B} - \sum_{n=1}^{\infty} n\bar{a}_n v^{n+1})(v-w)^{-1}dv \\ &= - \bar{B}rw^{-5} - \bar{B}E\bar{s}w^{-2} + r(\bar{s})^{-1} \sum_{n=1}^{3} n\bar{a}_n w^{n-4}. \end{aligned} \tag{63}$$

Applying Theorem III to the third integral, we have

$$(2\pi i)^{-1} \oint (v-w)^{-1} \bar{T}(\bar{v})\,dv$$

$$= (2\pi i)^{-1} \oint (v-w)^{-1} (\bar{s}B' v^{-1} - i\bar{s}S_{12} v^{-1} + \sum_{n=1}^{\infty} \bar{b}_n v^n)\,dv$$

$$= -\bar{s}(B' - iS_{12}) w^{-1}. \tag{64}$$

Adding the results of Eqs.(61), (63), and (64), since their sum is zero, we have

$$-\phi(w) + sBw - \bar{B}rw^{-5} - \bar{B}\bar{s}Ew^{-2} + r(\bar{s})^{-1} \sum_{n=1}^{3} n\bar{a}_n w^{n-4}$$

$$- \bar{s}(B' - iS_{12}) w^{-1} = 0. \tag{65}$$

Equation (65) determines $\phi(w)$ except for the constants $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$. These coefficients are found by multiplying Eq.(52) successively by $(2\pi i(v - w))^{-1}\, v\, dv$, $(2\pi i(v - w))^{-1}\, v^2\, dv$, and $(2\pi i(v - w))^{-1}\, v^3\, dv$, and repeating the integration around K. We have

$$\oint \frac{v\phi(v)\,dv}{2\pi i(v-w)} + \oint \frac{vH(v)\bar{\phi}'(\bar{v})\,dv}{2\pi i(v-w)} + \oint \frac{v\bar{T}(\bar{v})\,dv}{2\pi i(v-w)} = 0. \tag{66}$$

Inspection of Eq.(64) shows

$$\oint \frac{v\bar{T}(\bar{v})\,dv}{2\pi i(v-w)} = \oint \frac{v^2\bar{T}(\bar{v})\,dv}{2\pi i(v-w)} = \oint \frac{v^3\bar{T}(\bar{v})\,dv}{2\pi i(v-w)} = 0. \tag{67}$$

Inspection of Eq.(61) shows

$$\oint \frac{v\phi(v)\,dv}{2\pi i(v-w)} = -\sum_{n=2}^{\infty} a_n w^{1-n} = -w\phi(w) + sBw^2 + a_1.$$

Inspection of Eq.(63) shows

$$\oint \frac{vH(v)\bar{\phi}'(\bar{v})dv}{2\pi i(v-w)} = -\bar{B}rw^{-4} - \bar{B}\bar{s}Ew^{-1} + r(\bar{s})^{-1}\sum_{n=1}^{2} n\bar{a}_n w^{n-3}.$$

Upon substitution of these values, Eq.(66) reduces to

$$-w\phi(w) + sBw^2 + a_1 - \bar{B}rw^{-4} - \bar{B}\bar{s}Ew^{-1} + \frac{r}{\bar{s}}\sum_{n=1}^{2} n\bar{a}_n w^{n-3} = 0. \quad (68)$$

Multiplying Eq.(52) by $(2\pi i(v-w))^{-1} v^2\,dv$, and applying the results of Eq.(67), we have

$$\oint \frac{v^2\phi(w)dw}{2\pi i(v-w)} + \oint \frac{v^2H(v)\bar{\phi}'(\bar{v})dv}{2\pi i(v-w)} = 0. \qquad (69)$$

The value of the first integral is easily seen from Eq.(61). We have

$$\oint \frac{v^2\phi(v)dv}{2\pi i(v-w)} = -\sum_{n=3}^{\infty} a_n w^{2-n} = -w^2\phi(w) + sBw^3 + a_1 w + a_2.$$

The second integral is easily evaluated with the aid of Eq.(63). We have

$$\oint (2\pi i(v-w))^{-1} v^2 H(v)\bar{\phi}'(\bar{v})dv = -\bar{B}rw^{-3} + r(\bar{s})^{-1}\bar{a}_1 w^{-1}.$$

Hence, Eq.(69) reduces to

$$-w^2\phi(w) + sBw^3 + a_1 w + a_2 - \bar{B}rw^{-3} + r(\bar{s})^{-1}\bar{a}_1 w^{-1} = 0. \qquad (70)$$

Multiplying Eq.(52) by $(2\pi i(v - w))^{-1}\, v^3\, dv$, and applying the results of Eq.(67), we have

$$\oint (2\pi i(v - w))^{-1}\, v^3\, \phi(v)\, dv$$
$$+ \oint (2\pi i(v - w))^{-1}\, v^3 H(v)\bar{\phi}'(\bar{v})dv = 0. \qquad (71)$$

Inspecting Eq.(61), we see

$$\oint (2\pi i(v - w))^{-1}\, v^3\, \phi(v)\, dv$$
$$= - \sum_{n=4}^{\infty} a_n w^{3-n} = - w^3\phi(w) + sBw^4 + a_1 w^2 + a_2 w + a_3.$$

Inspecting Eq.(63), we see

$$\oint (2\pi i(v - w))^{-1} v^3 H(v)\bar{\phi}'(\bar{v})\, dv = - \bar{B}rw^{-2}.$$

Hence Eq.(71) becomes

$$- w^3\phi(w) + sBw^4 + a_1 w^2 + a_2 w + a_3 - \bar{B}rw^{-2} = 0. \qquad (72)$$

Let us rewrite Eqs.(65), (68), (70), and (72).

$$r\bar{a}_1(\bar{s})^{-1}w^{-3} + 2r\bar{a}_2(\bar{s})^{-1}w^{-2} + 3r\bar{a}_3(\bar{s})^{-1}w^{-1}$$

$$= \phi(w) - sBw + r\bar{B}w^{-5} + \bar{B}\bar{s}Ew^{-2} + \bar{s}(B' - iS_{12})w^{-1}. \quad (73)$$

$$a_1 + r\bar{a}_1(\bar{s})^{-1}w^{-2} + 2r\bar{a}_2(\bar{s})^{-1}w^{-1}$$

$$= w\phi(w) - sBw^2 + r\bar{B}w^{-4} + \bar{B}\bar{s}Ew^{-1}. \quad (74)$$

$$a_1w + r\bar{a}_1(\bar{s})^{-1}w^{-1} + a_2$$

$$= w^2\phi(w) - sBw^3 + r\bar{B}w^{-3}. \quad (75)$$

$$a_1w^2 + a_2w + a_3$$

$$= w^3\phi(w) - sBw^4 + r\bar{B}w^{-2}. \quad (76)$$

Multiplying Eq.(73) by w and subtracting from Eq.(74), we obtain

$$a_1 - 3r\bar{a}_3(\bar{s})^{-1} = -\bar{s}(B' - iS_{12}). \quad (77)$$

Multiplying Eq.(75) by w and subtracting from Eq.(76), we obtain

$$a_3 = r\bar{a}_1(\bar{s})^{-1}, \text{ or } \bar{a}_3 = \bar{r}a_1s^{-1}. \quad (78)$$

Substituting the value of $\bar{a}_3$ into Eq.(77), we have

$$a_1 = s(\bar{s})^2(B' - iS_{12})(3r\bar{r} - s\bar{s})^{-1}. \quad (79)$$

From Eqs.(78) and (79) we obtain the value of a_3.

$$a_3 = rs\bar{s}(B' + iS_{12})(3r\bar{r} - s\bar{s})^{-1}. \qquad (80)$$

Multiplying Eq.(74) by w and subtracting from Eq.(75),

$$a_2 - 2r\bar{a}_2(\bar{s})^{-1} = - \bar{B}\bar{s}E,$$

which gives

$$a_2 = (4r\bar{r} - s\bar{s})^{-1} (2rs^2 B\bar{E} + s(\bar{s})^2\bar{B}E), \qquad (81)$$

and $\phi(w)$ is completely determined. Setting $a_4 = 0$, and $a_5 = - r\bar{B}$, from Eq.(76) we have

$$\phi(w) = sBw + \sum_{n=1}^{5} a_n w^{-n}. \qquad (82)$$

To determine T(w) we multiply Eq.(53) by $(2\pi i(v - w))^{-1}dv$, and integrate around K.

$$\oint \frac{\bar{\phi}(\bar{v})\,dv}{2\pi i(v-w)} + \oint \frac{\bar{H}(\bar{v})\phi'(v)dv}{2\pi i(v-w)} + \oint \frac{T(v)\,dv}{2\pi i(v-w)} = 0. \qquad (83)$$

Performing the first integration with the aid of Theorem III, we obtain

$$\oint \frac{\bar{\phi}(\bar{v})\,dv}{2\pi i(v-w)} = \oint \frac{(\bar{s}\bar{B}w^{-1} + \sum_{n=1}^{5} \bar{a}_n w^n)}{2\pi i(v - w)}\, dv = - \bar{s}\bar{B}w^{-1}.$$

Performing the second integration with the aid of Theorem IV, we obtain

$$\oint \frac{\bar{H}(\bar{v})\emptyset'(v)dv}{2\pi i(v-w)} = -\bar{H}(\bar{w})\emptyset'(w) + \sum_{p=0}^{N} A_p w^p,$$

where we need to calculate the values of A_p, which are the coefficients of positive powers of w in the expansion of $\bar{H}(\bar{w})\emptyset'(w)$. We have

$$\bar{H}(\bar{w})\emptyset'(w) = \frac{(\bar{r}w^{11} + \bar{t}w^{8} + \bar{s}w^{5})}{(sw^{6} - 2tw^{3} - 5r)}\left(sB - \sum_{n=1}^{5} na_n w^{-n-1}\right)$$

$$= (\bar{r}s^{-1}w^5 + \bar{E}w^2 + H_o(w))(sB - \sum_{n=1}^{5} na_n w^{-n-1}),$$

where $H_o(w)$ consists of terms containing w with negative exponents. Noting that we may drop $H_o(w)$ and $a_5 w^{-6}$, we have

$$(\bar{r}s^{-1}w^5 + \bar{E}w^2)(sB - a_1w^{-2} - 2a_2w^{-3} - 3a_3w^{-4})$$

$$= \bar{r}Bw^5 - \bar{r}s^{-1}a_1w^3 + (Bs\bar{E} - 2\bar{r}s^{-1}a_2)w^2$$

$$- 3\bar{r}a_3s^{-1}w - \bar{E}a_1 - \bar{E}(2a_2w^{-1} + 3a_3w^{-2}).$$

The values of A_p are then obtained from the last expression.

$$\left.\begin{aligned}
A_0 &= -\bar{E}a_1. \\
A_1 &= -3\bar{r}a_3s^{-1}. \\
A_2 &= Bs\bar{E} - 2\bar{r}s^{-1}a_2 = -\bar{a}_2. \\
A_3 &= -\bar{r}s^{-1}a_1 = -\bar{a}_3. \\
A_4 &= a_4 = 0. \\
A_5 &= \bar{r}B = -\bar{a}_5. \\
a_1 &= s(\bar{s})^2(B' - iS_{12})(3r\bar{r} - s\bar{s})^{-1}. \\
a_2 &= (2rs^2B\bar{E} + s(\bar{s})^2\bar{B}E)(4r\bar{r} - s\bar{s})^{-1}. \\
a_3 &= rs\bar{s}(B' + iS_{12})(3r\bar{r} - s\bar{s})^{-1}. \\
a_4 &= 0. \\
a_5 &= -r\bar{B}. \\
B &= \bar{s}(S_1 + S_2)(4s')^{-1}. \\
B' &= (S_2 - S_1)/2. \\
E &= (\bar{s})^{-2}(t\bar{s} + 2\bar{t}r).
\end{aligned}\right\}(84)$$

Applying Theorem III to the third integral in Eq. (83), we have

$$\oint \frac{T(v)\,dv}{2\pi i(v-w)} = \oint \frac{(sB'v + isS_{12}v + \sum_{n=1}^{\infty} b_n v^{-n})}{(2\pi i(v - w)}\,dv =$$

$$= -\sum_{n=1}^{\infty} b_n w^{-n} = s(B' + iS_{12})w - T(w).$$

Substituting the last three integrals into Eq.(83), we have

$$T(w) = s(B' + iS_{12})w - \bar{s}\bar{B}w^{-1} - \bar{H}(\bar{w})\phi'(w) + \sum_{p=0}^{5} A_p w^p, \quad (85)$$

where

$$\bar{H}(\bar{w}) \equiv (\bar{s} + \bar{t}w^3 + \bar{r}w^6)(sw^6 - 2tw^3 - 5r)^{-1}w^5. \quad (86)$$

Substituting the values of $\phi(w)$ and $T(w)$ from Eqs.(82) and (85) into Eq.(33), we have

$$\begin{aligned} U = {} & \left[\bar{r}(\bar{w})^{-5} + \bar{t}(\bar{w})^{-2} + \bar{s}\bar{w}\right](sBw + \sum_{n=1}^{5} a_n w^{-n}) \\ & + \int_{w_0}^{w} \Big\{ s(B' + iS_{12})w - \bar{s}\bar{B}w^{-1} + \sum_{p=0}^{5} A_p w^p \\ & - (\bar{s} + \bar{t}w^3 + \bar{r}w^6)(sw^6 - 2tw^3 - 5r)^{-1}(sBw^5 - \sum_{n=1}^{5} na_n w^{4-n}) \Big\} \\ & \cdot (s - 2tw^{-3} - 5rw^{-6})dw, \quad (87) \end{aligned}$$

where the real part of the expression on the right is to be taken. The value of the constants can be obtained from Eqs.(84).

CHAPTER IV

THE STRESS COMPONENTS

In the preceeding chapter we found the stress function U in terms of w. With the aid of the results obtained, we can now express the stress components in terms of w. From Eqs.(39) and (48), we have

$$\widehat{xx} + \widehat{yy} = 4\,\mathrm{Re}(\phi'(z)) = 4\,\mathrm{Re}\left(\frac{\phi'(w)}{F'(w)}\right) \tag{88}$$

$$= 4\,\mathrm{Re}\left[\frac{\bar{F}'(\bar{w})\phi'(w)}{\bar{F}'(\bar{w})F'(w)}\right] = 4M^{-1}\mathrm{Re}(\bar{F}'(\bar{w})\phi'(w)),$$

where

$$M \equiv M(w) \equiv \bar{F}'(\bar{w})F'(w), \tag{89}$$

and M is real. From Eqs.(40), (49), and (57), we have

$$\frac{1}{2}(\widehat{yy} - \widehat{xx} + 2i\widehat{xy}) = \bar{z}\phi''(z) + T'(z) \tag{90}$$

$$= \frac{\bar{F}(\bar{w})F'(w)\phi''(w) - \bar{F}(\bar{w})F''(w)\phi'(w)}{(F'(w))^3} + \frac{T'(w)}{F'(w)}$$

$$= \frac{\bar{F}(\bar{w})F'(w)\phi''(w) - \bar{F}(\bar{w})\phi'(w)F''(w) + F'(w)F'(w)T'(w)}{(\bar{F}'(\bar{w}))^{-3}M^3}\,.$$

Let us denote the real part of $\bar{F}'(\bar{w})\phi'(w)$ by J_o. From Eq. (88)

$$\widehat{xx} + \widehat{yy} = 4M^{-1}J_o. \tag{91}$$

Let us denote the real part of $(\bar{F}'(\bar{w}))^3$ by J_2, and the imaginary part by J_{22}. If we denote the real part of $\bar{F}(\bar{w})F'(w)\phi''(w) - \bar{F}(\bar{w})\phi'(w)F''(w) + T'(w)F'(w)F'(w)$ by J_1, and the imaginary part by J_{11}, then Eq.(90) gives

$$\begin{aligned} \widehat{yy} - \widehat{xx} + 2i\widehat{xy} &= 2M^{-3}(J_1 + iJ_{11})(J_2 + iJ_{22}) \\ &= 2M^{-3}(J_1J_2 - J_{11}J_{22}) + 2iM^{-3}(J_1J_{22} + J_2J_{11}). \end{aligned} \tag{92}$$

From Eqs.(91) and (92), we have

$$\widehat{xx} = 2M^{-1}J_o - M^{-3}(J_1J_2 - J_{11}J_{22}), \tag{93}$$

$$\widehat{yy} = 2M^{-1}J_o + M^{-3}(J_1J_2 - J_{11}J_{22}), \tag{94}$$

and

$$\widehat{xy} = M^{-3}(J_1J_{22} + J_2J_{11}). \tag{95}$$

CASE ONE. Let $r = 0$, and let t and s be real.

Setting $w = Re^{i\beta}$, we have

$$M = s^2 - 4stR^{-3}\mathrm{Cos}\ 3\beta + 4t^2R^{-6},$$

$$\begin{aligned} J_o = {} & s^2B - 2stR^{-5}(S_{12}\mathrm{Sin}\ \beta + B'\mathrm{Cos}\ \beta) \\ & - s^2R^{-2}(S_{12}\mathrm{Sin}\ 2\beta - B'\mathrm{Cos}\ 2\beta) - 4t^2BR^{-6}, \end{aligned}$$

$$\begin{aligned} J_1 = {} & s^3B' + 2s^2tS_{12}R^{-5}(\mathrm{Sin}\ \beta + \mathrm{Sin}\ 7\beta) \\ & - 2s^2tB'R^{-5}(\mathrm{Cos}\ \beta + \mathrm{Cos}\ 7\beta) \\ & + 2sBR^{-2}(s^2 + 2t^2 - 6t^2R^{-4})\mathrm{Cos}\ 2\beta \\ & - 4s^2tR^{-3}(B'\ \mathrm{Cos}\ 3\beta + S_{12}\mathrm{Sin}\ 3\beta) \\ & + sR^{-4}(3s^2 + 6t^2 - 2t^2R^{-4} - 2s^2R^2)(B'\mathrm{Cos}\ 4\beta - S_{12}\mathrm{Sin}\ 4\beta) \\ & + 4tBR^{-5}(4t^2 + 2s^2 - 3s^2R^2)\mathrm{Cos}\ 5\beta \\ & + 4st^2R^{-6}(B'\mathrm{Cos}\ 6\beta + S_{12}\mathrm{Sin}\ 6\beta), \end{aligned}$$

$$J_2 = s^3 - 6s^2tR^{-3}\mathrm{Cos}\ 3\beta + 12st^2R^{-6}\mathrm{Cos}\ 6\beta - 8t^3R^{-9}\mathrm{Cos}\ 9\beta,$$

$$J_{22} = - 6s^2tR^{-3}\mathrm{Sin}\ 3\beta + 12st^2R^{-6}\mathrm{Sin}\ 6\beta - 8t^3R^{-9}\mathrm{Sin}\ 9\beta.$$

$$\begin{aligned}
J_{11} = {} & s^3S_{12} + 2s^2tB'R^{-5}(\mathrm{Sin}\ \beta + \mathrm{Sin}\ 7\beta) \\
& + 2s^2tS_{12}R^{-5}(\mathrm{Cos}\ \beta + \mathrm{Cos}\ 7\beta) \\
& - 2sBR^{-2}(s^2 + 2t^2 - 6t^2R^{-4})\mathrm{Sin}\ 2\beta \\
& - 4s^2tR^{-3}(S_{12}\mathrm{Cos}\ 3\beta - B'\mathrm{Sin}\ 3\beta) \\
& - sR^{-4}(3s^2 + 6t^2 - 2t^2R^{-4} - 2s^2R^2)(B'\mathrm{Sin}\ 4\beta + S_{12}\mathrm{Cos}\ 4\beta) \\
& - 4tBR^{-5}(4t^2 + 2s^2 - 3s^2R^2)\mathrm{Sin}\ 5\beta \\
& + 4st^2R^{-6}(S_{12}\mathrm{Cos}\ 6\beta - B'\mathrm{Sin}\ 6\beta).
\end{aligned}$$

These values of M, J_0, J_1, J_2, J_{11}, and J_{22}, when substituted into Eqs.(93), (94), and (95), enable us to calculate the stresses for Case I. The values of the stress components on the edge of the hole of Fig.1, Appendix I, were calculated in this manner. The results are given in Table 1, Appendix II. In table 2, Appendix II, the values of the stresses along the positive x-axis exterior to the hole are given. It will be seen that as x increases, the stresses rapidly approach their value at infinity.

The stress components for Case II and Case III can be obtained by following the procedure used for Case I.

CHAPTER V

SUMMARY

In this dissertation the author has made no attempt at exhausting the many types of holes obtainable from Eq. (4). The figures given in Appendix I represent some of the simpler holes obtained by equating certain parameters to zero. If the reader is interested in the stresses due to a certain hole included in Eq.(4), then any unnecessary parameters in that equation should be set equal to zero. In such cases some of the constants given in Eq.(84) will also be zero, which will simplify the computing of the stresses from Eqs.(93) to (95).

The stresses due to the hole given in Fig. 1, can be found in Appendix II. After the author computed these stresses, Dr. C. B. Smith called his attention to a similar hole

$$x = \cos 2\beta + 4 \cos \beta,$$

$$y = - \sin 2\beta + 4 \sin \beta,$$

for which A. E. Green has computed the stresses. His results are to be found in the Proceedings of the Royal Society of London, Vol. 184, pages 231-252. The method he used in solving his problem was quite different from the method employed here, and his boundary curve was

$$x = na\cos\beta + b\cos n\beta,$$

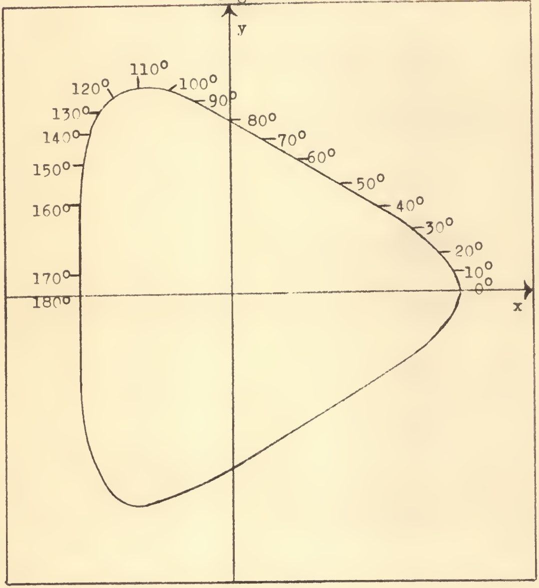
$$y = -na\sin\beta + b\sin n\beta.$$

In a paper of the same title as that given in reference 2 of the Bibliography, Martin Greenspan determined the stresses in a large, uniformly loaded plate weakened by a small hole of an approximately ovaloid shape. Greenspan employed a rather laborious method of piecing together particular solutions of the biharmonic equation for the stress function until all the boundary conditions could be satisfied. This process would become prohibitive in the case of more complicated boundary conditions. In reference 2 of the Bibliography, Morkovin applied the elegant and more general method of N. I. Mushelišvili for solving the same problem. Additional information on Mushelišvili's method may be found in references 7 and 8 of the Bibliography.

APPENDIX I

FIGURES OF THE HOLES

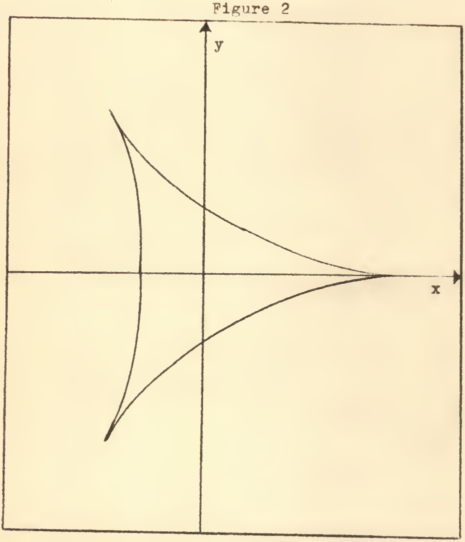
Figure 1



$$x = \text{Cos } 2\beta + 5 \text{ Cos } \beta$$

$$y = - \text{Sin } 2\beta + 5 \text{ Sin } \beta.$$

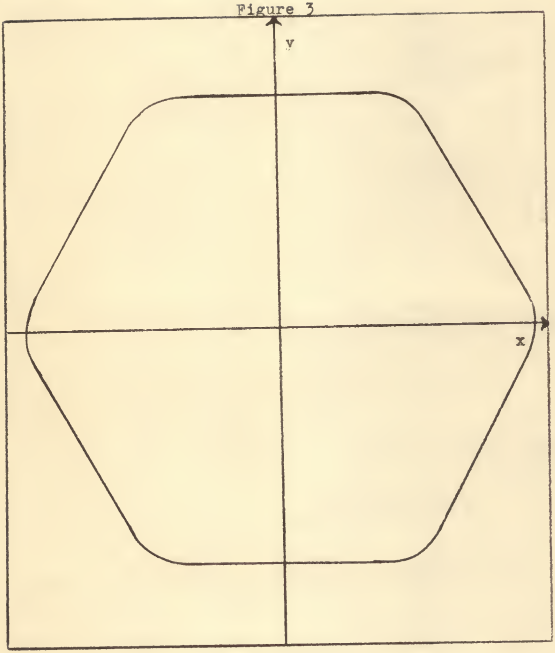
Figure 2



$$x = \cos 2\beta + 2\cos\beta$$

$$y = -\sin 2\beta + 2\sin\beta.$$

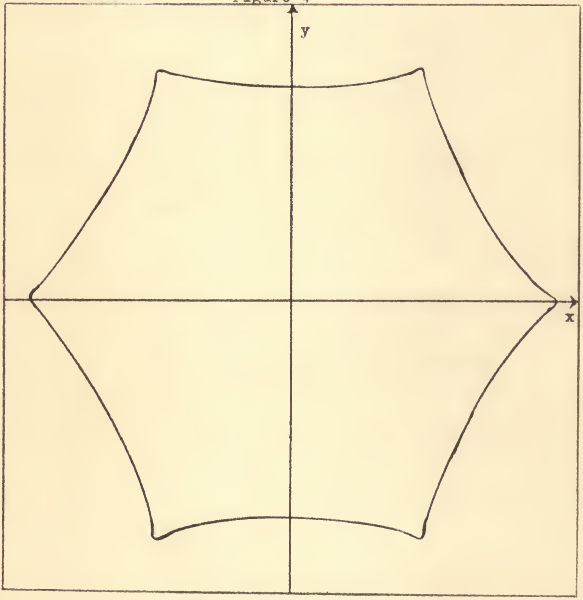
Figure 3



$$x = \cos 5\beta + 20 \cos \beta$$

$$y = -\sin 5\beta + 20 \sin \beta.$$

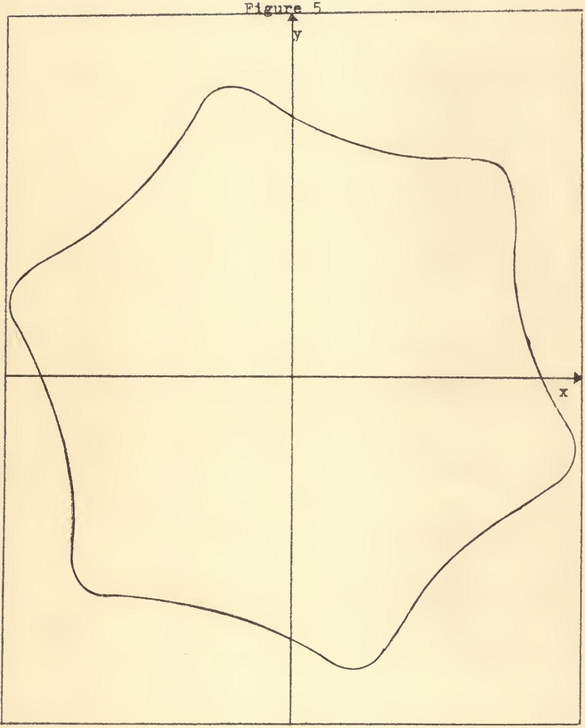
Figure 4



$$x = \text{Cos}\ 5\beta + 10\ \text{Cos}\ \beta$$

$$y = -\ \text{Sin}\ 5\beta + 10\ \text{Sin}\ \beta.$$

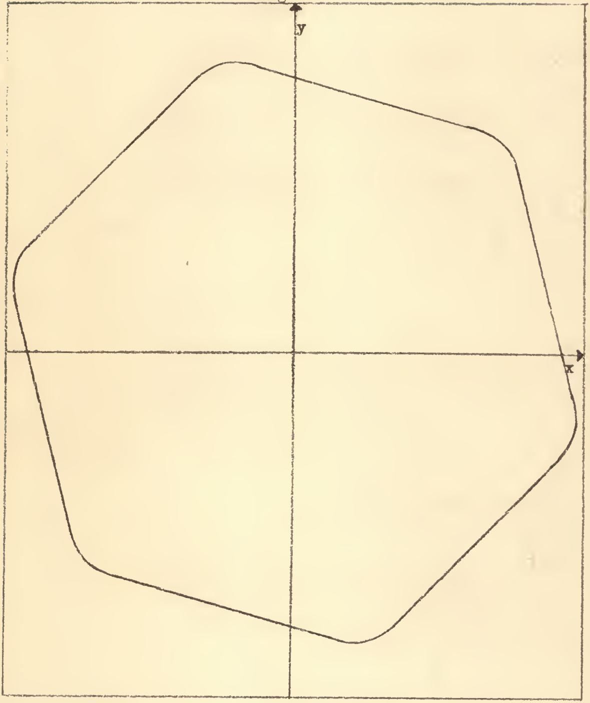
Figure 5



$$x = \mathrm{Cos}\ 5\beta + \mathrm{Sin}\ 5\beta + 10(\mathrm{Cos}\ \beta - \mathrm{Sin}\ \beta)$$

$$y = \mathrm{Cos}\ 5\beta - \mathrm{Sin}\ 5\beta + 10(\mathrm{Cos}\ \beta + \mathrm{Sin}\ \beta).$$

Figure 6



$x = \cos 5\beta + \sin 5\beta + 20(\cos \beta - \sin \beta)$

$y = \cos 5\beta - \sin 5\beta + 20(\cos \beta + \sin \beta).$

APPENDIX II

TABLES

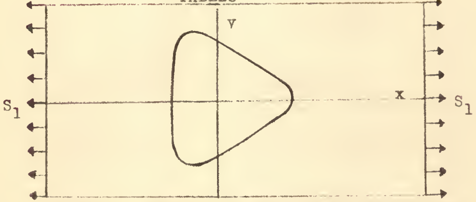


Table 1. Values of the stress components on the edge of the hole given in Figure 1.

β	$\widehat{xy}/S_1$	$\widehat{yy}/S_1$	$\widehat{xx}/S_1$
0°	0.000	-1.000	0.000
10°	.188	-.662	-.038
20°	-.132	-.206	.284
30°	-.214	.165	.294
40°	-.323	.220	.488
50°	-.410	.252	.666
60°	-.495	.286	.857
70°	-.673	.325	1.103
80°	-.724	.344	1.478
90°	-.844	.335	2.113
100°	-.709	.020	3.376
110°	.278	.052	4.440
120°	1.732	.268	3.732
130°	.822	1.094	.346
140°	.101	-.014	-.146
150°	-.104	-.745	-.041
160°	-.043	-.934	-.018
170°	-.026	-.986	-.0005
180°	0.000	-1.000	0.000

Table 2. Values of the stress components at points along the x-axis exterior to the hole given in Figure 1.

x	$\overset{\frown}{xx}/S_1$	$\overset{\frown}{yy}/S_1$
6.00	0.000	- 1.000
6.33	- .177	- .325
7.94	.192	.031
10.25	.543	.035
15.11	.770	.030
20.06	.871	.019
25.04	.910	.015
30.03	.925	.001

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BIOGRAPHY

The author was born at Colliers, West Virginia, on Dec. 7,1912. He attended the public schools at Wheeling, West Virginia, and graduated from Triadelphia High School. His undergraduate studies were pursued at the University of Miami, West Liberty College, Pennsylvania State College, and the University of Florida, where he graduated with the degree of Bachelor of Science in 1948. The author continued into graduate work in Mathematics and Physics at the University of Florida, and received the degree of Master of Science in 1950. He served as graduate assistant in the Department of Mathematics from 1948 to 1951, and as instructor in the C-2 Department from 1951 to 1952. He served three years with the Army in World War II.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences, and to the Graduate Council, and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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